TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 332, Number 2, August 1992

EXISTENCE OF POSITIVE NONRADIAL SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS IN ANNULAR DOMAINS

SONG-SUN LIN

ABSTRACT. We study the existence of positive nonradial solutions of equation $\Delta u + f(u) = 0$ in Ω_a, $u = 0$ on $\partial\Omega_a$, where $\Omega_a = \{x \in \mathbb{R}^n : a < |x| < 1\}$ is an annulus in $\mathbb{R}^n$, $n \geq 2$, and f is positive and superlinear at both 0 and ∞. We use a bifurcation method to show that there is a nonradial bifurcation with mode k at $a_k \in (0, 1)$ for any positive integer k if f is subcritical and for large k if f is supercritical. When f is subcritical, then a Nehari-type variational method can be used to prove that there exists $a^* \in (0, 1)$ such that for any $a \in (a^*, 1)$, the equation has a nonradial solution on Ω_a.

1. INTRODUCTION

In this paper we shall study the existence of positive nonradial solutions of the equation

$$\Delta u + f(u) = 0 \quad \text{in } \Omega_a, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega_a, \tag{1.2}$$

where $\Omega_a = \{x \in \mathbb{R}^n : a < |x| < 1\}$ is an annulus in $\mathbb{R}^n$, $n \geq 2$, and f satisfies the following conditions:

$$f \in C^1(\mathbb{R}^1) \quad \text{and} \quad f(u) > 0 \quad \text{for } u > 0, \tag{H-0}$$

$$f(0) = 0 \quad \text{and} \quad \lim_{u \to 0} f(u)/u = 0, \tag{H-1}$$

$$\liminf_{u \to \infty} u f'(u)/f(u) > 1. \tag{H-2}$$

This paper is motivated by the work of Brezis and Nirenberg [3] and Coffman [4]. In [3], Brezis and Nirenberg proved that for any fixed domain Ω_a, if $f(u) = u^p$ and $p < (n+2)/(n-2)$, $n \geq 3$, and is near to it, then (1.1) and (1.2) has a positive nonradial solution. Later on, in [4], Coffman studied (1.1), (1.2) with $f(u) = -u + u^p$, $p > 1$ and $n = 2$. He proved that the number of rotationally nonequivalent positive solutions grows without bound as $a \to 1^-$. In both papers, problems are subcritical and variational methods are used.

In this paper, we shall use two approaches to study the problems: the bifurcation method and the variational method.

Received by the editors July 25, 1989 and, in revised form, May 21, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35B32, 35J65, 35P30; Secondary 34B15, 34B25.
Key words and phrases. Nonradial solution, bifurcation method, variational method.
Work partially supported by the National Science Council of the Republic of China.

In applying the bifurcation method, we shall take a (the inner radius) as a bifurcation parameter. In spherical coordinates, the linearized equation of equations (1.1) and (1.2) at positive radial solution u_a is

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) + \left\{ f'(u_a) - \frac{k(k+n-2)}{r^2} \right\} \varphi(r) = -\mu_{k,l}(u_a)\varphi(r), \qquad a < r < 1, \tag{1.3}$$

$$\varphi(a) = 0 = \varphi(1), \tag{1.4}$$

where k and l are positive integers. It is well known that if there is a nonradial bifurcation at u_a, then $\mu_{k,l}(u_a) = 0$ for some k and l. Therefore, to look for $\mu_{k,l} = 0$, it is worth knowing the signs of $\mu_{k,1}(u_a)$ as a approaches to 1 or 0. We shall show that the condition (H-2) implies that, for any positive integer k, $\mu_{k,1}(u_a) < 0$ as a approaches 1. On the other hand, if "u_a tends to a positive radial solution u_0 of (1.1), (1.2) on the unit ball," then $\mu_{1,1}(u_a) > 0$ as a approaches 0.

Hence, if f is subcritical, i.e., f satisfies

$$\text{(H-3)} \quad \text{for } u \text{ large}, \quad f(u) \le \begin{cases} cu^p \text{ for some } p < \dfrac{n+2}{n-2} & \text{if } n \ge 3, \\ \exp A(u) \text{ with } A(u) = o(u^2) \text{ at } \infty & \text{if } n = 2, \end{cases}$$

then, for any $k \ge 1$, a nonradial bifurcation would occur at some $a_k \in (0, 1)$.

On the other hand, if f is supercritical, i.e., f satisfies

$$\text{(H-4)} \qquad uf'(u) \ge \frac{n+2}{n-2} f(u) \quad \text{for } u > 0,$$

we shall apply the McLeod-Serrin identity to show that $\mu_{1,1}(u_a) \ne 0$ for any u_a. For such f, we can prove that $\mu_{k,1}(u_a) > 0$ if k is large enough. Therefore a nonradial bifurcation would also occur at some $a_k \in (0, 1)$ when k is large.

For the subcritical case, a Nehari-type variational method will also be used to study the existence of positive nonradial solutions. Indeed, consider the functionals

$$J(v) = \int_{\Omega_a} \frac{1}{2}|\nabla v|^2 - F(v), \tag{1.5}$$

$$I(v) = \int_{\Omega_a} |\nabla v|^2 - vf(v), \tag{1.6}$$

on $H_0^1(\Omega_a)$, where $F(v) = \int_0^v f(t)\,dt$, and the numbers

$$j(a) = \inf\{J(v) : v \in H_0^1(\Omega_a) \text{ and } I(v) = 0\}, \tag{1.7}$$

and

$$j_\infty(a) = \inf\{J(v) : v \in H_0^1(\Omega_a),\ I(v) = 0 \text{ and } v \text{ is radial}\}. \tag{1.8}$$

If the minimizers of $j(a)$ and $j_\infty(a)$ are achieved with

$$j(a) < j_\infty(a), \tag{1.9}$$

then the minimizers of $j(a)$ will be nonradial, positive solutions of (1.1), (1.2).

For $a \in (0, 1)$, we can obtain (1.9) provided that all positive radial solutions of (1.1), (1.2) are "unstable with respect to nonradial modes," i.e., if u_a is a positive radial solution of (1.1) and (1.2), then there exists an eigenvalue $\mu_{k,1}(u_a) < 0$ for some positive integer k. Therefore, nonradial solutions exist, provided a is close to 1.

Existence and/or uniqueness of positive radial solutions of (1.1) and (1.2) have been studied by many authors, see, e.g., Ni and Nussbaum [17], Bandle, Coffman and Marcus [1], Garaizar [5] and Lin [11]. In case $f(0) > 0$, the existence of nonradial solutions has been studied by Suzuki and Nagasaki [21], Suzuki [22] and Lin [10, 12].

The paper is organized as follows: In §2, we briefly discuss some properties of positive radial solutions. In §3, we study the linearized equations (1.3) and (1.4) as $a \to 0^+$ or $a \to 1^-$. In §4, we use the McLeod-Serrin identity to study $\mu_{k,1} = 0$. In §5, an argument of degree theory is used to show that nonradial bifurcation actually occurs at u_a that satisfies $\mu_{k,1}(u_a) = 0$ and some appropriate conditions. In §6, a Nehari-type variational method is used to show that there exists a nonradial solution if f is subcritical and the annuli are narrow enough.

2. Radial solutions

In this section, we shall discuss some properties of positive radial solutions of (1.1) and (1.2) which will be used later.

A radial solution $u = u(r)$ of (1.1) and (1.2) satisfies the following equations

$$u''(r) + \frac{n-1}{r}u'(r) + f(u(r)) = 0, \qquad r \in (a, 1), \tag{2.1}$$

$$u(a) = 0 = u(1). \tag{2.2}$$

In Lin [11], it was proved that for any $a \in (0, 1)$, (2.1) and (2.2) have a positive radial solution u_a provided that f satisfies (H-0), (H-1) and

$$\lim_{u\to\infty} f(u)/u = \infty. \tag{H-2$'$}$$

It is clear that (H-2) implies (H-2)$'$. Therefore, if f satisfies (H-0) $\sim$ (H-2), then for any $a \in (0, 1)$, (2.1) and (2.2) have at least one positive radial solution.

For $n \geq 3$, set $s = r^{2-n}$ and $w(s) = u(r)$, then (2.1) and (2.2) can be written as

$$w''(s) + \rho(s)f(w(s)) = 0 \quad \text{in } (s_0, s_1), \tag{2.3}$$

$$w(s_0) = 0 = w(s_1), \tag{2.4}$$

where $\rho(s) = (n-2)^{-2}s^{-2-\varepsilon}$, $\varepsilon = 2/(n-2)$, $s_0 = 1$ and $s_1 = a^{2-n}$. For $n = 2$, set

$$s = -\log r \quad \text{and} \quad w(s) = u(r),$$

then equations (2.1), (2.2) can also be written as (2.3), (2.4) with $\rho(s) = e^{-2s}$, $s_0 = 0$ and $s_1 = -\log a$.

It is easy to check that solution w of (2.3) also satisfies the following integral equation

$$w(s) = w(\bar{s}) + w'(\bar{s})(s - \bar{s}) + \int_{\bar{s}}^{s} (t - s)\rho(t)f(w(t))\,dt \tag{2.5}$$

for $s, \bar{s} \in (s_0, s_1)$.

We first study radial solutions u_a when a is close to 1.

Proposition 2.1. *If u_a is a solution of* (2.1) *and* (2.2), *then we have*
(i) $\|u_a\|_\infty \to \infty$ *as* $a \to 1^-$,
(ii) $\int_{\Omega_a} |\nabla u_a|^2 \to \infty$ *as* $a \to 1^-$.

Proof. Let $w(s, \alpha)$ be the solution of the following initial value problem

$$w''(s) + \rho(s) f(w(s)) = 0, \qquad s > s_0, \tag{2.6}$$

$$w(s_0) = 0 \quad \text{and} \quad w'(s_0) = \alpha > 0. \tag{2.7}$$

Set $s_1(\alpha) = \sup\{\bar{s}\colon w(s, \alpha) > 0 \text{ in } (s_0, \bar{s})\}$, we claim that if $s_1(\alpha_j) \to s_0$ as $j \to \infty$, then $\alpha_j \to \infty$. We first prove that for any $\bar{s}_1 > s_0$, there exists $\delta > 0$ such that for any $\alpha \in (0, \delta)$,

$$w(s, \alpha) > 0 \quad \text{in } (s_0, \bar{s}_1]. \tag{2.8}$$

In fact, if $w(s, \alpha) > 0$ in $(s_0, \bar{s}_1)$, by (2.5), we have $w(s, \alpha) \le \alpha \bar{s}_1$ for $s \in [s_0, \bar{s}_1]$. Now, w satisfies

$$w''(s) + \rho(s) \frac{f(w(s))}{w(s)} w(s) = 0.$$

By (H-1) and the Sturm Comparison Theorem, (2.8) follows.

Next, we show that for any $0 < m < M$,

$$\inf\{s_1(\alpha)\colon \alpha \in [m, M]\} > s_0. \tag{2.9}$$

If (2.9) were false, then there would be a sequence $\{\alpha_j\} \subset [m, M]$ such that $\alpha_j \to \alpha_0 > 0$ and $s_1(\alpha_j) \to s_1(\alpha_0) = s_0$, a contradiction. This proves (2.9). Therefore, if $s_1(\alpha_j) \to s_0$ as $j \to \infty$, then $\alpha_j \to \infty$.

For large α, let $\tau(\alpha) \in (s_0, s_1)$ such that $w(\tau(\alpha), \alpha) = \|w(\cdot, \alpha)\|_\infty$. Then by the same argument as in Lemma 2.1 of [11], we have $\lim_{\alpha\to\infty} w(\tau(\alpha), \alpha) = \infty$. This proves (i).

(ii) Let $\tau(a) \in (a, 1)$ such that $\|u_a(\cdot)\|_\infty = u_a(\tau(a))$. Then

$$\begin{aligned} u_a(\tau(a)) = \int_a^{\tau(a)} u'(s)\,ds &\le (\tau(a) - a)^{1/2} \left\{ \int_a^{\tau(a)} u'(r)^2\,dr \right\}^{1/2} \\ &\le (\tau(a) - a)^{1/2} a^{(1-n)/2} \omega_n^{-1/2} \left\{ \int_{\Omega_a} |\nabla u_a|^2 \right\}^{1/2}, \end{aligned}$$

where ω_n is the area of unit sphere S^{n-1}. Hence, (ii) follows.

This completes the proof.

Next, we shall study radial solutions u_a when a is close to 0. Let $u_\alpha \equiv u(\cdot, \alpha)$ be the solution of (2.1), (2.2) with $a = a(\alpha) \in (0, 1)$, and

$$u'(1, \alpha) = -\alpha < 0. \tag{2.10}$$

It is easy to check that there exists a unique $\tau(\alpha) \in (a(\alpha), 1)$ such that $u(\tau(\alpha), \alpha) = \|u(\cdot, \alpha)\|_\infty$. For such u_α, define

$$\tilde{u}_\alpha(r) = \tilde{u}(r, \alpha) = \begin{cases} u(r, \alpha) & \text{if } r \in [\tau(\alpha), 1], \\ u(\tau(\alpha), \alpha) & \text{if } r \in [0, \tau(\alpha)]. \end{cases} \tag{2.11}$$

Note that u_0 is a positive radial solution of (1.1), (1.2) on the unit ball Ω_0 if it satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + f(u(r)) = 0, \qquad r \in (0, 1), \tag{2.12}$$

$$u'(0) = 0 = u(1). \tag{2.13}$$

Proposition 2.2. *Assume*

(i) $u_0 = u(\cdot, \alpha_0)$ *is a positive radial solution on the unit ball,*

(ii) *there is* $\delta > 0$ *such that for any* $\alpha \in (\alpha_0, \alpha_0 + \delta)$ (*or* $(\alpha_0 - \delta, \alpha_0)$), $u_\alpha \equiv u(\cdot, \alpha)$ *is a positive radial solution on the annulus with* $a = a(\alpha) \in (0, 1)$ *such that*

$$\|u_\alpha\|_\infty \le M < \infty. \tag{2.14}$$

Then, $\tilde{u}_\alpha$ *converges uniformly to* u_0 *on* $[0, 1]$ *as* $\alpha \to \alpha_0$.

Proof. Let $\tau(\alpha) \in (a(\alpha), 1)$ such that $u(\tau(\alpha), \alpha) = \|u_\alpha\|_\infty$. We first prove that

$$\lim_{\alpha \to \alpha_0} \tau(\alpha) = 0. \tag{2.15}$$

If (2.15) were false, there would be a sequence $\alpha_j \to \alpha_0$ and $\tau(\alpha_j) \to \tau_0 > 0$. Since $u'(\tau(\alpha_j), \alpha_j) = 0$, by the continuous dependence of o.d.e.'s, we have $u'(\tau_0, \alpha_0) = 0$. Since $u(\cdot, \alpha_0)$ is a solution on ball, by the result of Gidas, Ni and Nirenberg [6], $u'(r, \alpha_0) < 0$ on $(0, 1)$, a contradiction. This proves (2.15).

Denote

$$F(u) = \int_0^u f(s)\, ds \tag{2.16}$$

and define

$$V(r) \equiv V(r, \alpha) \equiv \tfrac{1}{2}u'^2(r) + F(u(r)). \tag{2.17}$$

Since

$$V'(r) = -\frac{n-1}{r}u'^2(r) < 0,$$

by (2.14), we have

$$\tfrac{1}{2}u'^2(r, \alpha) \le F(u(\tau(\alpha), \alpha)) \le M_1 < \infty \tag{2.18}$$

for all $r \in [\tau(\alpha), 1]$, where M_1 is a constant. Therefore, (2.11) and (2.18) imply that

$$|\tilde{u}'_\alpha(r)| \le (2M_1)^{1/2} \quad \text{on } [0, 1].$$

Hence, by the Ascoli-Arzela Theorem, there exists a $\tilde{u} \in C([0, 1])$ such that $\tilde{u}_\alpha \to \tilde{u}$ uniformly on $[0, 1]$ as $\alpha \to \alpha_0$. On the other hand, by (2.15), for any $r \in (0, 1)$, we have $u(r, \alpha) \to u_0(r)$ as $\alpha \to \alpha_0$. Hence, $\tilde{u} = u_0$ on $[0, 1]$.

The proof is complete.

It is not clear whether or not (2.14) always holds in Proposition 2.2. Here, we give some sufficient conditions which imply (2.14).

Proposition 2.3. *If*

(i) $n \geq 3$ *and there exists* $\delta > 0$ *such that*

$$\frac{n}{n-2} f(u) \geq f'(u)u \geq (1+\delta) f(u) \quad \text{for } u > 0, \tag{2.19}$$

or

(ii)

$$f(u) = u^p, \quad 1 < p < (n+2)/(n-2), \quad \text{if } n \geq 3 \text{ and } p \text{ is finite if } n = 2, \tag{2.20}$$

then there exists $\alpha_0 > 0$ *such that* $u(\cdot, \alpha_0)$ *is the unique solution on the ball and for any* $\alpha \in (\alpha_0, \infty)$, $u(\cdot, \alpha)$ *is the unique solution on the annulus* $(a(\alpha), 1)$. *Moreover, there exists* $M < \infty$, *such that for any* $\alpha \in (\alpha_0, \alpha_0 + 1)$, (2.14) *holds.*

Proof. By Theorems 1.2 and 1.4 of Ni and Nussbaum [17], we have the first part of the theorem. By Theorem 6.6 of Bandle et al. [1], there exists a unique positive radial solution for (2.1) with the boundary condition $u'(a) = 0 = u(1)$. Finally, by Theorems VII and IX of Nehari [14], (2.14) holds.

The proof is complete.

Remark 2.4. In [2], Bandle and Peletier proved that if $f(u) = u^{(n+2)/(n-2)}$, then $\|u_a\|_\infty \to \infty$ as $a \to 0^+$.

3. Linearized eigenvalue problems

To study the existence of nonradial solutions using bifurcation method, we need to investigate the linearized eigenvalue problem of (1.1), (1.2) at positive radial solutions u_a:

$$\Delta v + f'(u_a) v = -\mu v \quad \text{in } \Omega_a, \tag{3.1}$$

$$v = 0 \quad \text{on } \partial\Omega_a. \tag{3.2}$$

In spherical coordinates, (3.1), (3.2) are reduced to

$$\begin{aligned} \varphi''(r) + \frac{n-1}{r}\varphi'(r) + \left\{ f'(u_a) - \frac{\alpha_k}{r^2} \right\} \varphi(r) \\ = -\mu_{k,l}(u_a)\varphi(r), \qquad a < r < 1, \end{aligned} \tag{3.3}$$

$$\varphi(a) = 0 = \varphi(1), \tag{3.4}$$

where $\alpha_k = k(k+n-2)$, k and l are positive integers. Note that α_k are the eigenvalues of Laplacian $-\Delta$ on S^{n-1}, the unit sphere, and the dimension of the eigenspace $S_{n,k}$ of associated eigenfunctions is

$$l_{n,k} = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$

Let $\overline{x} = (x_1, \ldots, x_{n-1})$. A function v defined on S^{n-1} or Ω_a is called $O(n-1)$-invariant if $v(T\overline{x}, x_n) = v(\overline{x}, x_n)$ for all $T \in O(n-1)$. Then, for any positive integer k, the dimension of $V_{n,k} = \{v \in S_{n,k} | v \text{ is } O(n-1)\text{-invariant}\}$ is one, for details see [19].

We first prove that if f satisfies (H-2), then for any positive integer k, $\mu_{k,1}(u_a) < 0$ when a is close to 1.

Lemma 3.1. *If* (H-0) $\sim$ (H-2) *are satisfied. Then, for any* $k \geq 1$, *we have*

$$\lim_{a\to 1^-} \mu_{k,1}(a) = -\infty. \tag{3.5}$$

Proof. It is well known that $\mu_{k,1}$ can be characterized as

$$\mu_{k,1}(u_a) = \inf_{\psi\in X_a} Q_k(\psi)/I_2(\psi) \tag{3.6}$$

where

$$Q_k(\psi) \equiv Q_{k,a}(\psi) \equiv \int_a^1 r^{n-1}\left\{\psi'^2 - f'(u_a)\psi^2 + \frac{\alpha_k}{r^2}\psi^2\right\} dr, \tag{3.7}$$

$$I_2(\psi) \equiv I_{2,a}(\psi) \equiv \int_a^1 r^{n-1}\psi^2\, dr, \tag{3.8}$$

and $X_a = H_0^1((a, 1))$.

If u_a is a positive radial solution of (1.1), (1.2), then

$$\int_{\Omega_a} |\nabla u_a|^2 = \int_{\Omega_a} u_a f(u_a). \tag{3.9}$$

By (H-2), there exist $\varepsilon > 0$ and $M > 0$ such that

$$f'(u)u \geq (1+\varepsilon)f(u) \quad \text{for } u \geq M. \tag{3.10}$$

By (3.9), (3.10) and Proposition 2.1, we have

$$\begin{aligned}
\omega_n Q_k(u_a) &= \omega_n \int_a^1 r^{n-1}\left\{u_a'^2 - f'(u_a)u_a^2 + \frac{\alpha_k}{r^2}u_a^2\right\} dr \\
&= \int_{\Omega_a} \{u_a f(u_a) - f'(u_a)u_a^2\} + \alpha_k \int_{\Omega_a} u_a^2 r^{-2} \\
&\leq -\varepsilon \int_{\Omega_a} u_a f(u_a) + \alpha_k \int_{\Omega_a} u_a^2 r^{-2} + \int_{u_a\leq M} u_a f(u_a) - f'(u_a)u_a^2 \\
&\leq -\varepsilon \int_{\Omega_a} |\nabla u_a|^2 + \alpha_k a^{-2} \int_{\Omega_a} u_a^2 + M_1,
\end{aligned}$$

for some constant $M_1 \geq 0$.

Let $\nu_1(a)$ be the least eigenvalue of $-\Delta$ on Ω_a with the Dirichlet boundary condition. Then, it is easy to check that

$$\lim_{a\to 1^-} \nu_1(a) = \infty. \tag{3.11}$$

Using the Poincaré inequality

$$\int_{\Omega_a} |\nabla v|^2 \geq \nu_1(a) \int_{\Omega_a} v^2 \tag{3.12}$$

for all $v \in H_0^1(\Omega_a)$, we obtain

$$\omega_n Q_k(u_a) \leq \{-\varepsilon + \alpha_k a^{-2}\nu^{-1}(a)\} \int_{\Omega_a} |\nabla u_a|^2 + M_1.$$

Therefore, by using (3.11) and (3.12) again, (3.5) follows.

The proof is complete.

Next, we prove that if annulus solutions u_a tend to a solution u_0 on the ball, in the sense of Proposition 2.2, then $\mu_{1,1}(u_a) > 0$ as a approaches 0.

Lemma 3.2. *Under the hypotheses of Proposition* 2.2. *Then for any positive integer* k,

$$\lim_{\alpha\to\alpha_0}\mu_{k,1}(u_\alpha)=\mu_{k,1}(u_0)>0. \tag{3.13}$$

Proof. We first prove that $\mu_{1,1}(u_0)>0$. Since $u_0(r)$ satisfies (2.12), (2.13) with $u_0'(r)<0$ in $(0, 1)$, then v $(=-u_0')$ satisfies

$$v''(r)+\frac{n-1}{r}v'(r)+\left\{f'(u_0)-\frac{n-1}{r^2}\right\}v=0 \quad \text{in } (0, 1), \tag{3.14}$$

$$v(0)=0 \quad \text{and} \quad v>0 \text{ in } (0, 1). \tag{3.15}$$

Therefore, by using the Sturm Comparison Theorem, we have $\mu_{1,1}(u_0)>0$. Hence, $\mu_{k,1}(u_0)>0$ for any positive integer k.

Next, we shall divide the proof of (3.13) into two parts:

(i) $\limsup_{\alpha\to\alpha_0}\mu_{k,1}(u_\alpha)\le\mu_{k,1}(u_0)$,

(ii) $\liminf_{\alpha\to\alpha_0}\mu_{k,1}(u_\alpha)\ge\mu_{k,1}(u_0)$.

(i) Let $\psi_0>0$ be the eigenfunction associated with $\mu_{k,1}(u_0)$, i.e., $Q_{k,0}(\psi_0) = \mu_{k,1}(\psi_0)$ with the normalization $I_{2,0}(\psi_0)=1$. Define $\psi_\alpha\colon (a, 1)\to\mathbb{R}^1$, by $\psi_\alpha(r)=\psi_0((r-a)/(1-a))$, where $a=a(\alpha)\in(0, 1)$.

Then

$$\begin{aligned} Q_{k,a}(\psi_\alpha) &= \int_a^1 r^{n-1}\left\{\psi_\alpha'^2 - f'(u_\alpha)\psi_\alpha^2+\frac{\alpha_k}{r^2}\psi_\alpha^2\right\}dr \\ &= Q_1(\alpha)+Q_2(\alpha)+Q_3(\alpha), \end{aligned}$$

where

$$\begin{aligned} Q_1(\alpha) = \int_0^1 &\{(1-a)^{-2}\psi_0'^2(t)-f'(u_0)\psi_0^2(t) \\ &+\alpha_k[a+(1-a)t]^{-2}\psi_0^2(t)\}[a+(1-a)t]^{n-1}(1-a)\,dt, \end{aligned}$$

$$Q_2(\alpha)=\int_0^1 r^{n-1}\{f'(u_0)-f'(\tilde u_\alpha)\}\psi_0^2\left(\frac{r-a}{1-a}\right)dr,$$

and

$$Q_3(\alpha)=\int_a^{\tau(a)} r^{n-1}\{f'(\tilde u_\alpha)-f'(u_\alpha)\}\psi_0^2\left(\frac{r-a}{1-a}\right)dr.$$

By Proposition 2.2 and (2.15), for any $\varepsilon>0$, we have $Q_{k,a}(\psi_\alpha)\le\mu_{k,1}(u_0)+\varepsilon$ when α is sufficiently close to α_0.

This proves (i).

(ii) Let ψ_α be the eigenfunction associated with $\mu_{k,1}(u_\alpha)$ and $I_{2,a}(\psi_\alpha)=1$. Define

$$\overline{\psi}_\alpha(r)=\begin{cases}\psi_\alpha(r) & \text{if } r\in[a, 1],\\ 0 & \text{if } r\in[0, a].\end{cases}$$

Then

$$\begin{aligned} Q_{k,0}(\overline{\psi}_\alpha) &= \int_0^1 r^{n-1}\left\{\overline{\psi}_\alpha'^2-f'(u_0)\overline{\psi}_\alpha^2+\frac{\alpha_k}{r^2}\overline{\psi}_\alpha^2\right\}dr \\ &= \int_a^1 r^{n-1}\left\{\psi_\alpha'^2-f'(u_\alpha)\psi_\alpha^2+\frac{\alpha_k}{r^2}\psi_\alpha^2\right\}dr \\ &\quad+\int_a^1 r^{n-1}\{f'(u_\alpha)-f'(u_0)\}\psi_\alpha^2\,dr \\ &= \mu_{k,1}(u_\alpha)+Q_4(\alpha)+Q_5(\alpha), \end{aligned}$$

where

$$Q_4(\alpha) = \int_{\tau(\alpha)}^{1} r^{n-1}\{f'(u_\alpha) - f'(u_0)\}\psi_\alpha^2\, dr$$

and

$$Q_5(\alpha) = \int_{a(\alpha)}^{\tau(\alpha)} r^{n-1}\{f'(u_\alpha) - f'(u_0)\}\psi_\alpha^2\, dr.$$

We claim that

$$\lim_{\alpha\to\alpha_0} \int_{a(\alpha)}^{\tau(\alpha)} r^{n-1}\psi_\alpha^2(r)\, dr = 0. \tag{3.16}$$

Since u_α are uniformly bounded, it is easy to check that $\mu_{k,1}(u_\alpha)$ are bounded, say,

$$|\mu_{k,1}(u_\alpha)| \le C_1, \tag{3.17}$$

for some constant $C_1 > 0$. Therefore, by (3.6), (3.7), and (3.17), we obtain $\int_{\Omega_a} |\nabla\psi_\alpha|^2 \le C_2$, for some constant $C_2 > 0$. By the Sobolev Imbedding Theorem, we have $\int_\Omega \psi_\alpha^{2n/(n-2)} \le C_3$ for some constant $C_3 > 0$. Finally, by the Hölder inequality, we have

$$\int_a^\tau r^{n-1}\psi_\alpha^2 \le \left\{\int_a^\tau r^{n-1}\right\}^{2/n} \left\{\int_a^\tau \psi_\alpha^{2n/(n-2)}\right\}^{(n-2)/n} \le C_4\tau(\alpha)^2$$

for some constant $C_4 > 0$. This proves (3.16). By (3.16) and Proposition 2.2, we have

$$\lim_{\alpha\to\alpha_0} Q_4(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to\alpha_0} Q_5(\alpha) = 0.$$

This proves (ii), and the proof is complete.

Definition 3.3. Let u_a, $a \in (0, 1)$, be a family of positive radial solutions of (1.1) and (1.2). u_a is called smooth in a if u_a is continuous in a with respect to the L^∞ norm.

Definition 3.4. A smooth family of positive radial solutions u_a is said to converge to a positive radial solution u_0 on unit ball if $\tilde{u}_a$ converges uniformly to u_0 on $[0, 1]$ as $a \to 0^+$, where $\tilde{u}_a$ is as in (2.11).

Theorem 3.5. *Assume* (H-0) ~ (H-2) *are satisfied. Let* u_a, $a \in (0, 1)$, *be a smooth family of positive radial solutions of* (1.1) *and* (1.2) *which converges to a positive radial solution* u_0 *on the unit ball as* $a \to 0^+$. *Then, for any* $k \ge 1$, *there exists* $a_k \in (0, 1)$ *such that*

$$\mu_{k,1}(u_{a_k}) = 0. \tag{3.18}$$

Proof. The result follows from Lemmas 3.1 and 3.2 and the continuous dependence of eigenvalue $\mu_{k,1}$ on u_a.

Theorem 3.6. *If* f *satisfies* (2.19) *or* (2.20), *then for any* $k \ge 1$, *there exists* $a_k \in (0, 1)$ *such that* $\mu_{k,1}(u_{a_k}) = 0$. *Moreover, we have*

$$\mu_{k,l}(u_{a_k}) > 0 \tag{3.19}$$

for all integers $l \ge 2$.

Proof. By Proposition 2.3 and Theorem 3.5, we obtain the first part of the results. Since the unique radial solutions u_a, $a \in (0, 1)$, can be obtained by a Nehari-type variational method. By Lemma 6.3 of Bandle et al. [1],

$$\mu_{0,l}(u_a) \geq 0 \tag{3.20}$$

for all integers $l \geq 2$. Hence (3.19) follows from (3.20).

The proof is complete.

For supercritical case, we have the following result.

Theorem 3.7. *Assume* (H-0) $\sim$ (H-2) *are satisfied. Let* u_a, $a \in (\delta, 1)$ *and* $\delta \geq 0$, *be a smooth family of positive radial solutions of* (1.1) *and* (1.2). *Then, for sufficiently large* k, *there exists* $a_k \in (\delta, 1)$ *such that* $\mu_{k,1}(u_{a_k}) = 0$.

Proof. For a fixed $a \in (\delta, 1)$, there exists $k_0 \geq 1$ such that for any $k \geq k_0$, we have $\alpha_k/r^2 \geq f'(u_a)$ on $[a, 1]$. Hence, $\mu_{k,1}(u_a) > 0$ for $k \geq k_0$. Therefore, the result follows from Lemma 3.1.

The proof is complete.

4. McLeod-Serrin identity

In this section we shall use the McLeod-Serrin identity [13] to study $\mu_{k,1} = 0$.

We first recall the identity and let u and ψ satisfy

$$u'' + \frac{n-1}{r}u' + f(u) = 0 \tag{4.1}$$

and

$$\psi'' + \frac{n-1}{r}\psi' + g(r)\psi = 0, \tag{4.2}$$

respectively. Let

$$Y = r^{a-b}\psi, \qquad Z = \{r^a(u-c)\}', \qquad W = YZ' - ZY'$$

and $D = (r^m w)'$, where $m = n - 1 - 2a + 2b$ and a, b and c are constants, then

$$\begin{aligned} \frac{D}{r^{m-2}Y} &= \{(b-1)(b+n-3) + r^2[g(r) - f'(u)]\}Z \\ &\quad + 2r^a(b-1)(a-n+2)u' \\ &\quad + ar^{a+1}\left\{(u-c)f'(u) - \left(1 + \frac{2b}{a}\right)f(u)\right\}. \end{aligned} \tag{4.3}$$

If we choose $c = 0$ and $g(r) = f'(u) - \alpha/r^2$, then (4.3) can be written as

$$(r^m W)' = \{Ar^{n-4+b}u + Br^{n-3+b}u' + r^{n-2+b}C(u)\}\psi, \tag{4.4}$$

where

$$A = a\{(b-1)(b+n-3) - \alpha\}, \tag{4.5}$$

$$B = (b-1)(b+n-3) - \alpha + 2(b-1)(a-n+2), \tag{4.6}$$

$$C(u) = auf'(u) - (a+2b)f(u) \tag{4.7}$$

and

$$\begin{aligned} r^m W &= a(b-1)r^{n-3+b}u\psi + (a+b+1-n)r^{n-2+b}u'\psi \\ &\quad - ar^{n-2+b}u\psi' - r^{n-1+b}u'\psi' - r^{n-1+b}f(u)\psi. \end{aligned} \tag{4.8}$$

Furthermore, if we choose a and b such that $B = 0$, i.e.,

$$(b-1)(b+n-3) - \alpha + 2(b-1)(a-n+2) = 0, \tag{4.9}$$

and assume that u and ψ satisfy

$$u(R) = 0 = u(1) \quad \text{and} \quad \psi(R) = 0 = \psi(1), \tag{4.10}$$

where $R \in (0, 1)$. Then, by integrating (4.4) from R to 1, we have

$$\begin{aligned} R^{n-1+b}u'(R)\psi'(R) - A\int_R^1 r^{n-4+b}u\psi - \int_R^1 r^{n-2+b}C(u)\psi \\ = u'(1)\psi'(1). \end{aligned} \tag{4.11}$$

By choosing appropriate a and b, we can now prove the following result for supercritical f.

Theorem 4.1. *Assume f satisfies*

$$uf'(u) \geq \frac{n+2}{n-2}f(u) \quad \textit{for } u > 0. \tag{H-4}$$

If u_R is a solution of (2.1) *and* (2.2) *with $a = R$, then $\mu_{1,1}(u_R) \neq 0$.*

Proof. Suppose $\mu_{1,1}(u_R) = 0$. Let ψ be an associated eigenfunction with $\psi > 0$ in $(R, 1)$. Since $\alpha = \alpha_1 = n-1$, it is easy to check that a $(= 0)$ and b $(= 0)$ satisfy (4.9) and thus $A = 0$ and $C(u) = 0$. Therefore, (4.11) becomes

$$R^{n-1}u'(R)\psi'(R) = u'(1)\psi'(1). \tag{4.12}$$

On the other hand, a $(= n-2)$ and b $(= 2)$ satisfy (4.9) too. For these choices, we have $A = 0$ and $C(u) = (n-2)uf'(u) - (n+2)f(u)$. Therefore, (4.11) becomes

$$\begin{aligned} R^{n+1}u'(R)\psi'(R) - \int_R^1 r^n\{(n-2)uf'(u) - (n+2)f(u)\}\psi \\ = u'(1)\psi'(1). \end{aligned} \tag{4.13}$$

Therefore, if f satisfies (H-4), then (4.12) and (4.13) lead to a contradiction.
The proof is complete.

Corollary 4.2. *For $n \geq 3$ and $p \geq (n+2)/(n-2)$, let u_R be the unique positive radial solution of*

$$u'' + \frac{n-1}{r}u' + u^p = 0 \quad \textit{in } (R, 1), \tag{4.14}$$

$$u(R) = 0 = u(1). \tag{4.15}$$

Then $\mu_{1,1}(u_R) < 0$ and $\mu_{k,l}(u_R) > 0$ for $k \geq 1$ and $l \geq 2$.

Proof. By Theorem 4.1, $\mu_{1,1}(u_R) \neq 0$. By Lemma 3.1, $\mu_{1,1}(u_R) < 0$ for R close to 1. Hence $\mu_{1,1}(u_R) < 0$ for all $R \in (0, 1)$. By (3.20), we have $\mu_{k,l}(u_R) > 0$ for all $k \geq 1$ and $l \geq 2$.
The proof is complete.

Also the McLeod-Serrin identity has to do with $\mu_{k,1} \neq 0$ for $k \geq 2$. As an example, we prove the following results.

Theorem 4.3. *For $n \geq 3$ and $p > 1 + 2(k+1)/(n-2)$, let u_R be the solution of* (4.14) *and* (4.15). *Then $\mu_{k,1}(u_R) < 0$ if*

$$R \geq R(k, p, n) = \{2(k-1)/[(n-2)(p-1) - 2(k+1)]\}^{1/2k}. \tag{4.16}$$

Proof. For any fixed α, in equation (4.9), a can be solved in terms of b; in fact,

$$a = \frac{n-1-b}{2} + \frac{\alpha}{2(b-1)}, \quad \text{for } b \neq 1. \tag{4.17}$$

Let $\alpha = \alpha_k = k(k+n-2)$. Then

$$A = -2\{b-(k+1)\}\{b+(k+n-3)\}\{b-(n+k-1)\}\{b+k-1\}/(b-1)$$

and

$$C(u) = cu^p, \quad \text{where } c = a(p-1) - 2b.$$

Choosing $b_1 = -(k-1)$ and $b_2 = (k+1)$, we have $A_1 = A_2 = 0$, $c_1 = 2(k-1)$ and $c_2 = (n-2)(p-1) - 2(k+1)$. By (4.10) and (4.11), we have

$$\begin{aligned} &R^{n-k}u'(R)\psi'(R) - c_1 \int_R^1 r^{n-k-1}u^p\psi \\ &\quad = R^{n+k}u'(R)\psi'(R) - c_2 \int_R^1 r^{n+k-1}u^p\psi, \end{aligned}$$

which implies

$$c_1 \int_R^1 r^{n-k-1}u^p\psi > c_2 \int_R^1 r^{n+k-1}u^p\psi.$$

Therefore, if for any $r \in [R, 1]$

$$c_1 r^{n-k} \leq c_2 r^{n+k}, \tag{4.18}$$

then (4.10) does not hold, i.e., $\mu_{k,1}(u_R) \neq 0$. Finally, (4.16) follows from (4.18) by a straightforward computation.

The proof is complete.

Remark 4.4. For a fixed $k \geq 2$, $R(k, p, n) \to 0$ as $p \to \infty$. Therefore, it is of interest to know whether or not there exists a finite $p^*(k, n) > 0$ such that $\mu_{k,1}(u_R) < 0$ for all $R \in (0, 1)$ if $p > p^*(k, n)$. Note that $p^*(1, n) = (n+2)/(n-2)$.

5. Symmetry breaking

In this section, we shall study the problem of nonradial bifurcation (symmetry breaking) of (1.1) and (1.2) at a positive radial solution u_a with $\mu_{k,1}(u_a) = 0$, where $k \geq 1$.

To begin with, we shall take a as a bifurcation parameter, (i.e., we vary domains). As for handling these problems we shall work in the Lagrangian formulation and then in the Eulerian formulation for computational purpose (see, e.g., Henry [7]).

We begin with the Lagrangian formulation. Fix a constant $c \in (0, 1)$ and denote $\Omega = \Omega_c$. Then for any $t \in (0, 1)$, $\Omega_t = h_t(\Omega)$, where in spherical coordinates, h_t is given by

(5.1)

$$h_t(r, \theta_1, \ldots, \theta_{n-1}) = \left(1 + \frac{t-1}{c-1}(r-1), \theta_1, \ldots, \theta_{n-1}\right), \qquad r \in (c, 1).$$

The pull back $h_t^*: C^m(\Omega_t) \to C^m(\Omega)$ is defined by

$$w(y, t) \equiv (h_t^* u)(y) = u(h_t(y)), \qquad y \in \Omega. \tag{5.2}$$

Then, equations (1.1), (1.2) on Ω_t can be rewritten as

$$L_t w + f(w) = 0, \quad \text{in } \Omega, \tag{5.3}$$

$$w = 0, \quad \text{on } \partial\Omega, \tag{5.4}$$

where $L_t = h_t^* \Delta (h_t^*)^{-1}$. Moreover, (5.3) and (5.4) are equivalent to the nonlinear operator equation

$$w(\cdot, t) - \Phi_t(w(\cdot, t)) = 0 \tag{5.5}$$

on $C_0^{1+\delta}(\overline{\Omega}) \times (0, 1)$, where the nonlinear operator $\Phi_t: C_0^{1+\delta}(\overline{\Omega}) \times (0, 1) \to C_0^{1+\delta}(\overline{\Omega})$ is given by

$$\Phi_t(w) = \Phi(w, t) = (-L_t)^{-1} f(w), \tag{5.6}$$

$\delta \in (0, 1)$ is a constant.

Since Φ_t is a compact operator on $C_0^{1+\delta}(\Omega) \times [a, b]$, $[a, b] \subset (0, 1)$, the method of degree theory can be applied to equation (5.5).

On the other hand, in the Eulerian formulation, let $u(x, t)$ be a positive solution of (1.1), (1.2) on Ω_t, which is smooth in t. Let $v(x, t) = \partial u(x, t)/\partial t$. Then v satisfies the following linearized equations of (1.1) and (1.2) at u:

$$\Delta v(x, t) + f'(u(x, t)) v(x, t) = 0, \quad \text{in } \Omega_t, \tag{5.7}$$

$$v(x, t) + V(x, t) \cdot \nabla u(x, t) = 0, \quad \text{on } \partial\Omega_t, \tag{5.8}$$

where in spherical coordinates,

$$V(x, t) = \left(\frac{|x| - 1}{t - 1}, 0, \dots, 0 \right). \tag{5.9}$$

If $u(x, t) = u(|x|, t)$ is a positive radial solution of (1.1) and (1.2), let $v(x, t) = \varphi(r, t)\psi(\theta_1, \dots, \theta_{n-1})$. Then (5.7) and (5.8) are reduced to

$$\begin{gathered} \varphi''(r, t) + \frac{n-1}{r}\varphi'(r, t) + \left\{ f'(u) - \frac{\alpha_k}{r^2} \right\} \varphi(r, t) \\ = -\mu_{k,l}\varphi(r, t), \qquad r \in (t, 1), \\ \varphi(t, t) = 0 = \varphi(1, t), \end{gathered}$$

for $k \ge 0$ and $l \ge 1$. These equations have been studied in previous sections.

We need the following terminology:

Definition 5.1. Let u_t, $t \in (a_0, b_0) \subset (0, 1)$, be a smooth family of positive radial solutions of (1.1) and (1.2). $a \in (0, 1)$ is called a nonradial bifurcation point (with respect to u_t) if every neighborhood of (u_a, a) in $C_0^{1+\delta}(\overline{\Omega}) \times (0, 1)$ contains a nonradial positive solution of (1.1) and (1.2). If a is a bifurcation point and $\mu_{k,1}(u_a) = 0$, $k \ge 1$, then a is called a nonradial bifurcation point with mode k. Similarly, $[a, b] \subset (a_0, b_0)$ is called a nonradial bifurcation interval if every neighborhood of $\{(u_t, t), t \in [a, b]\}$ in $C_0^{1+\delta}(\overline{\Omega}) \times (0, 1)$ contains a nonradial positive solution of (1.1) and (1.2). In both cases, we say that u_t has a nonradial bifurcation (or symmetry breaking) on $(0, 1)$.

We shall restrict (5.5) on the $O(n-1)$-invariant subspace $\{w \in C_0^{1+\gamma}(\overline{\Omega}) \times (0, 1): w \text{ is } O(n-1)\text{-invariant}\}$, see the end of the first paragraph of §3. The following result is a variant of bifurcation theorems of Krasnosel'ski [9] or Rabinowitz [18], which was proved essentially in Lin [12]. The proof is omitted.

Theorem 5.2. *Let u_t be the family of positive radial solutions of* (1.1) *and* (1.2) *which are smooth in $t \in (a_0, b_0) \subset (0, 1)$.*

If $a \in (a_0, b_0)$ and there exist $\varepsilon > 0$ and $k \geq 1$ such that

(i) *$\mu_{k,1}(u_a) = 0$ and $\mu_{k,1}(u_t)\mu_{k,1}(u_{t'}) < 0$ for $t \in (a-\varepsilon, a)$ and $t' \in (a, a+\varepsilon)$,*

(ii) *$\mu_{k,2}(t) > 0$ for $t \in (a-\varepsilon, a+\varepsilon)$, then a is a nonradial bifurcation point with mode k.*

Similarly, if (i) *and* (ii) *are replaced by*

(i)′ *$\mu_{k,1}(u_t) = 0$ on $[a, b]$ and $\mu_{k,1}(u_t) \cdot \mu_{k,1}(u_{t'}) < 0$ for $t \in (a-\varepsilon, a)$ and $t' \in (b, b+\varepsilon)$,*

(ii)′ *$\mu_{k,2}(t) > 0$ for $t \in (a-\varepsilon, b+\varepsilon)$, then $[a, b]$ is a nonradial bifurcation interval.*

Theorem 5.3. *If f satisfies* (2.19) *or* (2.20), *then for any $k \geq 1$, the radial solution u_a has a nonradial bifurcation with mode k on $(0, 1)$. If $f(u) = u^p$ with $p \geq (n+2)/(n-2)$, then there exists $k^*(p) > 1$, such that for any $k \geq k^*(p)$, u_a has a nonradial bifurcation with mode k on $(0, 1)$.*

Proof. The results follow from Theorems 3.6, 3.7 and 5.2.

6. Variational method

In this section, we shall use the Nehari-type variational method to study the existence of positive nonradial solution of (1.1), (1.2).

Consider the functionals

$$J(v) = \int_{\Omega_a} \frac{1}{2}|\nabla v|^2 - F(v) \tag{6.1}$$

and

$$I(v) = \int_{\Omega_a} |\nabla v|^2 - v f(v) \tag{6.2}$$

on $H_0^1(\Omega)$, where $F(v) = \int_0^v f(t)\,dt$. Let

$$M = \{v \in H_0^1(\Omega_a) \colon I(v) = 0\}, \tag{6.3}$$

$$M_r = \{v \in M \colon v \text{ is radial}\}. \tag{6.4}$$

Let u_a be a positive radial solution of (1.1) and (1.2) which is unstable with respect to nonradial mode, i.e., the following conditions hold:

(U) there are eigenvalues $\mu_1 < \mu_2 < 0$ and eigenfunctions $v_1 = v_1(r) > 0$ and $v_2 = \varphi(r)\psi(\theta_1, \dots, \theta_{n-1})$ with $\varphi(r) > 0$ in $(a, 1)$ and $\psi \not\equiv 0$ such that

$$\Delta v_1 + f'(u_a)v_1 = -\mu_1 v_1 \quad \text{in } \Omega_a, \tag{6.5}$$

$$v_1 = 0 \quad \text{on } \partial\Omega_a, \tag{6.6}$$

and

$$\Delta v_2 + f'(u_a)v_2 = -\mu_2 v_2 \quad \text{in } \Omega_a, \tag{6.7}$$

$$v_2 = 0 \quad \text{on } \partial\Omega_a. \tag{6.8}$$

We first prove the following lemmas which generalize the results of Bandle et al. [1].

Lemma 6.1. *Assume* f *satisfies* (H-0), (H-1) *and*

(H-2)″ *there is* $\tau > 0$ *such that* $uf'(u) \geq (1+\tau)f(u)$ *for all* $u > 0$.

Let u_a *be a positive radial solution of* (1.1) *and* (1.2) *and satisfy* (U). *Then there exist an* $\varepsilon > 0$ *and a smooth function* $\delta\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^1$ *with* $\delta(0) = \delta'(0) = 0$ *such that for any* $t \in (-\varepsilon, \varepsilon)$,

$$I(u_a + \delta(t)v_1 + tv_2) = 0. \tag{6.9}$$

Proof. Define the function $H(\delta, t)\colon \mathbb{R}^2 \to \mathbb{R}^1$ by $H(\delta, t) = I(u_a + \delta v_1 + tv_2)$. Then, it is easy to verify that

$$H(\delta, 0) = \delta \int_{\Omega_a} \{f(u_a) - f'(u_a)u_a\}v_1 + O(\delta^2)$$

as $\delta \sim 0$. Hence,

$$\frac{\partial H}{\partial \delta}(0, 0) = \int_{\Omega_a} \{f(u_a) - f'(u_a)u_a\}v_1 < 0. \tag{6.10}$$

By the implicit function theorem, there exist $\varepsilon > 0$ and a smooth function $\delta\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^1$ with $\delta(0) = 0$ such that (6.9) holds. To show $\delta'(0) = 0$, we note that

$$\frac{\partial H}{\partial \delta}(\delta(t), t)\frac{d\delta}{dt} + \frac{\partial H}{\partial t}(\delta(t), t) = 0, \tag{6.11}$$

and as $t \sim 0$, we have

$$\begin{aligned} H(0, t) &= I(u_a + tv_2) \\ &= \int_{\Omega_a} |\nabla u_a|^2 + 2t\nabla u_a \cdot \nabla v_2 + t^2|\nabla v_2|^2 \\ &\quad - (u_a + tv_2)\left\{f(u_a) + f'(u_a)tv_2 + \frac{1}{2}f''(u_a)t^2v_2^2\right\} + O(t^3) \\ &= t\int_{\Omega_a} 2\nabla u_a \cdot \nabla v_2 - \{f(u_a) + u_a f'(u_a)\}v_2 \\ &\quad + t^2\int_{\Omega_a} |\nabla v_2|^2 - f'(u_a)v_2^2 - \frac{1}{2}f''(u_a)u_a v_2^2 + O(t^3) \\ &= t^2\int_{\Omega_a} \mu_2 v_2^2 - \frac{1}{2}f''(u_a)u_a v_2^2 + O(t^3), \end{aligned}$$

here,

$$\int_{S^{n-1}} \psi(\theta_1, \dots, \theta_{n-1}) = 0 \tag{6.12}$$

has been used repeatedly. Therefore, $\partial H(0, 0)/\partial t = 0$. By (6.11), we have $\delta'(0) = 0$.

The proof is complete.

Lemma 6.2. *Assume* (H-0), (H-1) *and* (H-2)″ *are satisfied. Let* u_a *be a positive radial solution of* (1.1) *and* (1.2) *and satisfy* (U). *Then*

$$J(u_a + \delta(t)v_1 + tv_2) = J(u_a) + \tfrac{1}{2}\mu_2 t^2 + O(t^4) \tag{6.13}$$

as $t \to 0$. *In particular,* $J(u_a)$ *is not the infimum of* J *over* M.

Proof. After some calculations, we have

$$\begin{aligned} &J(u_a + \delta(t)v_1 + tv_2) - J(u_a) \\ &\quad = \frac{1}{2}\mu_1\delta^2(t)\int_{\Omega_a} v_1^2 + \frac{1}{2}\mu_2 t^2 \int_{\Omega_a} v_2^2 + O(t^4), \end{aligned}$$

here (6.12) are used. Since $\delta(0) = \delta'(0) = 0$, (6.13) follows.

The proof is complete.

Now, we can prove the following theorem.

Theorem 6.3. *Assume* (H-0), (H-1), (H-2)$''$ *and* (H-3) *are satisfied. Then there exists an* $a^* \in (0, 1)$ *such that for any* $a \in (a^*, 1)$, (1.1) *and* (1.2) *have a nonradial solution.*

Proof. We note that (H-2)$''$ implies $M_1 = 0$ in the proof of Lemma 3.1. Therefore, by Lemma 3.1, there exists an $a^* \in (0, 1)$ such that $\mu_{1,1}(u_a) < 0$ for any positive radial solution u_a of (1.1) and (1.2) with $a \in (a^*, 1)$. Hence, by Lemma 6.2, we have $J(u_a) > j(a) \equiv \inf_{v \in M} J(v)$. Since $j(a)$ is achieved by some $\overline{u}_a \in M$ and $\overline{u}_a$ is a positive solution of (1.1) and (1.2), (see, e.g., Ni [16]). Therefore, $\overline{u}_a$ is nonradial.

The proof is complete.

References

1. C. Bandle, C. V. Coffman, and M. Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations **69** (1987), 322–345.
2. C. Bandle and L. A. Peletier, *Nonlinear elliptic problems with critical exponent in shrinking annuli*, Math. Ann. **280** (1988), 1–19.
3. H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **34** (1983), 437–477.
4. C. V. Coffman, *A nonlinear boundary value problem with many positive solutions*, J. Differential Equations **54** (1984), 429–437.
5. X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **70** (1987), 69–92.
6. B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
7. D. Henry, *Topics in nonlinear analysis*, Trabalho de Mathematica No. 192, Universidade de Brasilia, 1982.
8. B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math., vol. 1150, Springer, 1985, pp. 95–97.
9. M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, 1964.
10. S. S. Lin, *On non-radially symmetric bifurcation in the annulus*, J. Differential Equations **80** (1989), 251–279.
11. ——, *On the existence of positive radial solutions for nonlinear elliptic equations in annular domains*, J. Differential Equations **81** (1989), 221–233.
12. ——, *Positive radial solutions and non-radial bifurcation for semilinear elliptic equations in annular domains*, J. Differential Equations **86** (1990), 367–391.
13. K. McLeod and J. Serrin, *Uniqueness of positive radial solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **99** (1987), 115–145.
14. Z. Nehari, *On a class of nonlinear second order differential equations*, Trans. Amer. Math. Soc. **95** (1960), 101–123.

15. W.-M. Ni, *Some aspects of semilinear elliptic equations*, Lecture Notes, National Tsing Hua University, Hsinchu, Taiwan, May 1987.

16. ——, *Recent progress in semilinear elliptic equations*, Solutions for Nonlinear Elliptic Equations (T. Suzuki, ed.), RIMS Kokyuroku 679, Research Institute for Mathematical Science, Kyoto University, Kyoto, Japan, February, 1989.

17. W.-M. Ni and R. Nussbaum, *Uniqueness and non-uniqueness for positive radial solution of* $\Delta u + f(u, r) = 0$, Comm. Pure Appl. Math. **38** (1985), 67–108.

18. P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis **7** (1971), 487–513.

19. J. Smoller and A. Wasserman, *Symmetry-breaking for solution of semilinear elliptic equations with general boundary conditions*, Comm. Math. Phys. **105** (1986), 415–441.

20. T. Suzuki and K. Nagasaki, *On the nonlinear eignevalue problem* $\Delta u + \lambda e^u = 0$, Trans. Amer. Math. Soc. **309** (1988), 591–608.

21. ——, *Lifting of local subdifferentiations and elliptic boundary value problems on symmetric domains*. I, Proc. Japan Academy **64** (1988), 1–4.

22. T. Suzuki, *Symmetric domains and elliptic equations*, preprint.

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSIN-CHU, TAIWAN, REPUBLIC OF CHINA

E-mail address: sslin@twnctu01.bitnet